e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 2

Subject:Mathematics

Title/Heading of topic: Successive differentiation

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SUCCESSIVE DIFFERENTIATION

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), \dots, f^{(n)}(x)$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

$\vdots$

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

1.2 Calculation of n^{th} Derivatives

i. **n^{th} Derivative of e^{ax}**

Let $y = e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$\vdots$$
$$y_n = a^n e^{ax}$$

ii. **n^{th} Derivative of $(ax + b)^m$, m is a +ve integer greater than n**

Let $y = (ax + b)^m$

$$y_1 = m\,a(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax + b)^{m-2}$$
$$\vdots$$
$$y_n = m(m-1)\dots(m-n+1)a^n(ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax+\mathrm{b})^{m-n}$$

iii. n^{th} Derivative of $y = \log(ax+b)$

Let $y = \log(ax+b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$\vdots$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

iv. n^{th} Derivative of $y = \sin(ax+b)$

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$\vdots$

$$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax+b)$

$$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

v. n^{th} Derivative of $y = e^{ax}\sin(ax+b)$

Let $y = e^{ax}\sin(bx+c)$

$$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax} b\cos(bx+c)$$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[\,r\,cos\alpha \sin(bx+c) + r\,sin\alpha \cos(bx+c)]$$

Putting $a = r\,cos\alpha,\ b = r\,sin\alpha$

$$= e^{ax}\ r\ \sin(bx+c+\alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx+c+2\alpha)$

$\vdots$

$$y_n = e^{ax}\ r^n\ \sin(bx+c+n\alpha)$$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$

$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, m>0, m>n \\ 0, \quad m>0, \quad m<n, \\ n!\, a^n, \quad m=n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, \quad m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
$y = e^{ax}\sin(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$y = e^{ax}\cos(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow y_n = (-1)^{n+1}\, n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let $y = \sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10x + \cos 2x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x+\frac{n\pi}{2}\right) + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of $sin^2 x cos^3 x$

Solution: Let $y = sin^2 x cos^3 x$

$$= sin^2x cos^2x\ cosx$$
$$= \frac{1}{4} sin^2 2x\ cosx\ = \frac{1}{8}(1 - cos4x)cosx$$
$$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$$
$$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$$
$$= \frac{1}{16}(2cosx - cos3x - cos5x)$$
$$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$
$$= \left(\frac{1}{2} 2\ sin^2x\right)^2$$
$$= \frac{1}{4}((1 - \cos 2x)^2$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$$
$$= \frac{3}{8} - \frac{1}{2} cos2x + \frac{1}{8} cos4x$$
$$\therefore y_n = -\frac{1}{2} 2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} 4^n cos\left(4x + \frac{n\pi}{2}\right)$$

Example 5 Find the n^{th} derivative of $e^{3x} \cos x \sin^2 2x$

Solution: Let $y = e^{3x} \cos x \sin^2 2x$

Now $\cos x \sin^2 2x = \frac{1}{2} (\cos x - \cos x \cos 4x)$

$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x + \cos 3x\right)$$
$$\Rightarrow y = e^{3x} \cos x \sin^2 2x = \frac{1}{2} e^{3x} \cos x - \frac{1}{4} e^{3x} \cos 5x - \frac{1}{4} e^{3x} \cos 3x$$
$$\therefore y_n = \frac{1}{2} e^{3x} (9 + 1)^{\frac{n}{2}} \cos\left(x + n \tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x} (9 + 25)^{\frac{n}{2}} \cos\left(5x + n \tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x} (9 + 9)^{\frac{n}{2}} \cos\left(3x + n \tan^{-1}\frac{3}{3}\right)$$
$$= \frac{1}{2} e^{3x}\, 10^{\frac{n}{2}} \cos\left(x + n \tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\, 34^{\frac{n}{2}} \cos\left(5x + n \tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x}\, 18^{\frac{n}{2}} \cos(3x + n \tan^{-1} 1)$$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$
$$\therefore y_n = a^n \left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$$

$$= a^n \left[\left\{ \sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right) \right\}^2 \right]^{\frac{1}{2}}$$

$$= a^n \left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right) \right]^{\frac{1}{2}}$$

$$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$$

$$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\,]^{\frac{1}{2}}$$

$$= a^n[1 + (-1)^n \sin 2ax\,]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

Substituting $2x+1 = r\,cos\theta,\ \sqrt{3} = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\ r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\ [cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta]$$

$$\because \sqrt{3} = r\,sin\theta$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\left(\sqrt{3}\right)^{n+2}}\ 2i\,sin(n+1)\theta\ sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\ sin(n+1)\theta\ sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x\,(\sec x \tan x) = 2\ sec^2 x \tan x$$

$$\therefore\ \cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\,sec^2 x \tan x - 2(x + \tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2})\,\frac{dy}{dx} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2})\,\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$